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Random Composition of Two Rational Maps: Singularity of the Invariant Measure

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We study the invariant measure of a Markov chain obtained by randomly composing two rational maps related to the Anderson model with a Bernoulli potential. For a certain range of the parameters we show that the invariant measure is singular continuous. In certain cases the support turns out to be a Cantor set with a multifractal structure.

KEY WORDS: Markov chains; invariant measure; Anderson localization; fractals.

1. INTRODUCTION

This paper is concerned with the following problem, which arises in the mathematical theory of one-dimensional disordered systems. We refer the reader to the early works by Dyson and Schmidt and to the review by Ishii.[(1)]

Let $\{v_n\}$ be i.i.d. random variables with values in $\{0, 1\}$ and for $\lambda > 0$ let ϕ_λ be the solution of the Cauchy problem for the difference equation of the Schrödinger type:

$$[\lambda v(n) - E]\, \phi(n) = \phi(n+1) + \phi(n-1) \tag{1.1}$$

$$\phi(-1) = \alpha, \qquad \phi(0) = \beta$$

where α, β are real and $E \in \mathbf{R}$ has the physical interpretation of an energy. We introduce the random variables

$$z_n = \phi_\lambda(n-1)/\phi_\lambda(n)$$

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It is readily seen from (1.1) that the variables z_n obey the random recursion relation

$$z_n = 1/(\lambda v_n - E - z_{n-1}) \tag{1.2}$$

and thus they form a Markov chain in $\mathbf{R}$. If we denote by $p = \text{Prob}(v = 0)$ and by $\nu = \nu_{E,p,\lambda}$ the invariant measure of the chain [whose existence is a basic result of Furstemberg; see, e.g., Ref. 2, p. 30], then the question we raise is the following: under which condition on λ and E is the measure ν absolutely continuous with respect to the Lebesgue measure?

This particular problem is actually part of the more general question of the long-time behavior of a Markov chain obtained by randomly composing two (or more) maps in some metric space. This kind of stochastic process arises in different fields, such as the statistical mechanics of disordered magnets,[3] in discrete biological models,[4] or in the theory of random perturbations of dynamical systems.[5] In this work we will concentrate on the particular case described above where the maps are unimodular; in a subsequent paper we will consider more general situations. In our context very interesting but nonrigorous results have been obtained by Derrida and Gardner[6] by means of perturbation theory around the free case $\lambda = 0$; their results indicate a positive answer to the above problem for $\lambda \ll 1$ and E close to one the special values

$$E_{k,q} = 2\cos(\pi q/L), \qquad k, q \in \mathbf{N} \tag{1.3}$$

On the other hand, Pincus[7] showed that if each of the two rational maps that appear in (1.2),

$$T_0(x) = -1/E + x, \qquad T_1(x) = 1/\lambda - E - x \tag{1.4}$$

has two fixed points, one stable and the other unstable, then under some extra condition on E and λ, the support of the invariant measure becomes a Cantor set of zero Lebesgue measure. In this case ν becomes singular continuous, since by very general results ν cannot be pure point (Ref. 2, p. 30). Pincus' condition is, however, inadequate to discuss the mathematically interesting case when at least one of the two maps is elliptic. This occurs for E in the set $[-2, 2] \cup [\lambda - 2, \lambda + 2]$, that is, when E is in the spectrum of the infinite stochastic Jacobi matrix given by Eq. (1.1) looked upon as a bounded operator in $l^2(\mathbf{Z})$ (see, e.g., Ref. 8). From the physical point of view, it is precisely for these values of the energy E that the problem becomes relevant, since a detailed knowledge of the invariant measures of the variables z_n provides a great deal of information about the spectral properties of H (see, e.g., Ref. 2, Part B, Chapter 2).

In this paper we show that if λ is taken large enough, depending on p, but not on E, then the measure ν becomes singular continuous. In order to state our result in a precise way, let us introduce the Liapunov exponent given by the formula

$$\gamma(\lambda, E) = -\int d\nu_{\lambda,E}(z) \ln(|z|) \tag{1.5}$$

According to Furstemberg's basic result, γ is strictly positive for any $\lambda > 0$ and any $E \in \mathbf{R}$ and it expresses (with probability one) the rate of the exponential growth of the solution of (1.1) for a fixed initial condition. More precisely, one has

$$\gamma(\lambda, E) = \lim_n (1/n) \log |\phi_\lambda(n)| \tag{1.6}$$

for almost all the configurations $\{v_n\}$ in $\{0, 1\}^{\mathbf{Z}}$ with respect to the Bernoulli measure of parameter p.

With the above notations our main results can be expressed as follows:

Theorem 1. If $\gamma(\lambda, E) > \ln(2)/2$, then ν is singular continuous.

Corollary. If $\lambda > \exp[\ln(2)/2K(p)]$ with $K(p) = (1-p)^2/2[1 + (1-p)p^2]$, then ν is singular continuous.

The corollary is a simple consequence of Theorem 1 and of the following result due to Martinelli and Micheli[9]:

$$\inf \gamma(\lambda, E) > K(p) \ln(\lambda) \tag{1.7}$$

Remark 1. The above theorem is the exact analogue of a result proved by Carmona *et al.*[10] for the integrated density of states (ids) $N(E)$ of the random matrix H. As is well known, the ids $N(E)$ can be expressed in terms of the invariant measure by

$$N(E) = \int_{-\infty}^{0} d\nu_{\lambda,E}(z) \tag{1.8}$$

Remark 2. For general results concerning the relationship among Liapunov exponent, entropy, and dimension of the measure ν, see Ref. 11.

For the special values of the energy E in $[-2, 2]$ given in (1.3) with q and L relatively prime, it is possible to compute in a rather explicit way the support of the invariant measure at least for λ large. The reason is that for such energies the Lth power of the map T_0 becomes the identity and this simplifies the problem considerably. Our result is the following:

Theorem 2. Let $E=2\cos(\pi q/L)$, with q and L relatively prime integers. Then there exists a $\lambda_c(L)$ such that if $\lambda \geqslant \lambda_c(L)$, then the support of the invariant measure is contained in a Cantor set of zero Lebesgue measure.

One may ask at this point whether the critical value of λ given by Theorem 2 is such that Theorem 1 also applies, namely if $\gamma(\lambda_c, E) > \ln(2)/2$. We analyzed this problem for the special case $E=0$, namely $L=2$, and we found that $\lambda_c(2)=2$, while $\gamma(\lambda_c, E) < \ln(2)/2$ for any value of the probability parameter $0 < p < 1$. Thus, in this case Theorem 2 gives a more refined result.

In this particular situation we also analyzed numerically the structure of the support of the invariant measure and we found that it is a Cantor set with a multifractal structure. For this last part we followed the approach to multifractality suggested in Ref. 12.

2. PROOF OF THEOREM 1

In order to simplify the exposition, we first fix some useful notations. We denote by ω_L a sequence of 0's and 1's of length L, by $\omega_L(j)$ the number at the jth position in the sequence, $1 < j < L$, and by

$$P(\omega_L) = p^{\#\{j;\omega_L(j)=0\}}(1-p)^{\#\{j;\omega_L(j)=1\}}$$

its probability. Given a sequence ω_L, we can associate to it the rational map T_{ω_L} obtained by composing the maps T_0, T_1 in the order given by ω_L:

$$T_{\omega_L} = T_{\omega_L(L)} \circ \cdots \circ T_{\omega_L(2)} \circ T_{\omega_L(1)} \tag{2.1}$$

If we write

$$T_{\omega_L} = \frac{a_L x - b_L}{c_L x - d_L} \tag{2.2}$$

then we have the recursion relation

$$\begin{aligned} a_L &= c_{L-1} \\ b_L &= d_{L-1} \\ c_L &= [\lambda\omega_L(L) - E]\, c_{L-1} - a_{L-1} \\ dL &= [\lambda\omega_L(L) - E]\, d_{L-1} - b_{L-1} \end{aligned} \tag{2.3}$$

From (2.3) we also obtain the identity

$$a_L d_L - b_L c_L = a_{L-1} d_{L-1} - b_{L-1} c_{L-1} \tag{2.4}$$

which implies

$$a_L d_L - b_L c_L = 1 \tag{2.5}$$

Finally, we will denote by S the point $x = d_L/c_L$ where the map T_{ω_L} becomes singular.

It is now possible to explain in simple terms the idea behind the proof of our main theorem.

Using Fustemberg's result on the positivity of the Liapunov exponent $\gamma(\lambda, E)$, we will show that with large probability, the map T_{ω_L} will be almost flat with the exception of a tiny interval around the singularity, where it will be extremely steep with derivative $T'_{\omega_L} = o[\exp(2\gamma L)]$. This fact will imply that a large portion of the real line will be mapped by T_{ω_L} into a small interval of size $O[\exp(-2\gamma L)]$. Since the total number of these intervals as ω_L varies does not excede 2^L, if γ is as in the theorem, we have that with large probability the process z_L will lie in a set of vanishing Lebesgue measure as L tends to infinity, and the result will follows.

We now start with the technical details.

Lemma 1. For every $\varepsilon > 0$ the probability

$$P\left(\frac{d}{dx} T_{\omega_L}|_{x=0} < e^{-(2\gamma - \varepsilon)L}\right) \to 1$$

as $L \to +\infty$.

Proof. By direct computation

$$T'_{\omega_L}(0) = 1/d_L^2 \tag{2.6}$$

Since it is well known that

$$\lim_{L \to \infty} (1/L) \ln |d_L| = \gamma \qquad \text{a.s.} \tag{2.7}$$

(see, e.g., Ref. 2, p. 228), the lemma follows immediately.

Lemma 2. Suppose that

$$T'_{\omega_L}(0) < e^{-(2\gamma - \varepsilon)L}$$

then

$$T'_{\omega_L}(x) < \frac{e^{-(2\gamma - \varepsilon)L}}{(1 - x/S)^2}$$

Proof. Using (2.2) and (2.5), we get

$$T'_{\omega_L}(x) = \frac{1}{(c_L x - d_L)^2} = \frac{1}{d_L^2(1 - x/S)^2} \tag{2.8}$$

We are now ready to complete the proof of the theorem.

Let $\varepsilon > 0$, $k > 2$ and define for any ω_L the set

$$I^k_{\omega L} \equiv [-k, k] \qquad \text{if} \quad |S| > 2k$$

$$I^k_{\omega L} \equiv \mathbf{R} \backslash [S(1-\varepsilon), S(1+\varepsilon)] \qquad \text{if} \quad |S| \leqslant 2k$$

with ε small enough and k sufficiently large.

Next we construct a deterministic set $\Sigma = \Sigma(k, \varepsilon)$ of zero Lebesgue measure but with positive ν, provided k and ε are sufficiently large and sufficiently small, respectively. We set

$$\Sigma = \bigcap_i \bigcup_{L \geqslant i} \bigcup_{\omega_L \in \Omega_L} T_{\omega_L} I^k_{\omega_L} \tag{2.9}$$

where

$$\Omega_L = \{\omega_L; T'_{\omega_L}(x)|_{x=0} < e^{-(2\gamma - \varepsilon)L}\}$$

By Lemma 2 for any $\omega_L \in \Omega_L$ the Lebesgue measure of the interval $T_{\omega_L} I^k_{\omega_L}$ is smaller than

$$|T_{\omega_L} I^k_{\omega_L}| \leqslant 4 \frac{k}{\varepsilon} e^{-(2\gamma - \varepsilon)L} \tag{2.10}$$

and therefore if $2 \exp[-(2\gamma - \varepsilon) L] < 1$, the Lebesgue measure of Σ is zero since

$$|\Sigma| \leqslant \lim_{i \to \infty} \sum_{L \geqslant i} 4 \cdot 2^L (i/\varepsilon) \exp[-(2\gamma - \varepsilon) L] = 0 \tag{2.11}$$

Using Lemma 1, we will now show that we can choose k and ε such that $\nu(\Sigma) = 0$.

We have in fact

$$\begin{aligned} \nu(\Sigma) &\geqslant \lim_{L \to \infty} \nu\left(\bigcup_{\omega_L \in \omega_L} T_{\omega_L} I^k_{\omega_L}\right) \\ &\geqslant \lim_{L \to \infty} \sum_{\omega_L \in \Omega_L} P(\omega_L)\, \nu(I^k_{\omega_L}) \end{aligned} \tag{2.12}$$

This last inequality is a simple consequence of the equation expressing the invariance of the measure ν:

$$\nu(A) = p\nu(T_0^{-1}A) + (1-p)\,\nu(T_1^{-1}A) \tag{2.13}$$

for any $ACB(\mathbf{R})$.

If we take A as

$$A = \bigcup_{\omega_L \in \Omega_L} T_{\omega_L} I_{\omega_L}^k$$

we get

$$\begin{aligned}
\nu\left(\bigcup_{\omega_L \in \Omega_L} T_{\omega_L} I_{\omega_L}^k\right) &= (1-p)\,\nu(T_1^{-1} \bigcup_{\omega_L \in \Omega_L} T_{\omega_L} I_{\omega_L}^k) \\
&\quad + p\nu\left(T_0^{-1} \bigcup_{\omega_L \in \Omega_L} T_{\omega_L} I_{\omega_L}^k\right) \\
&\geqslant (1-p)\,\nu\left(\bigcup_{\omega_L \in \Omega_L, \omega_L(L)=1} T_{\omega_{L-1}} I_{\omega_L}^k\right) \\
&\quad + p\nu\left(\bigcup_{\omega_L \in \Omega_L, \omega_L(L)=0} T_{\omega_{L-1}} I_{\omega_L}^k\right)
\end{aligned} \tag{2.14}$$

By iterating (2.14) L times, we arrive at (2.12).

The rhs of (2.12) can now be bounded from below by

$$\begin{aligned}
&\lim_{L\to\infty} \sum_{\omega_L \in \Omega_L} P(\omega_L)\,\nu(I_{\omega_L}^k) \\
&\quad \geqslant \lim_{L\to\infty} P(\Omega_L) \min(\nu(-k,k), \inf_{|\eta|<2k} \nu(\mathbf{R}\backslash[\eta(1-\varepsilon), \eta(1+\varepsilon)]) \\
&\quad \geqslant \min\{\nu(-k,k), \inf_{|\eta|<2k} [1-\nu(\eta(1-\varepsilon), \eta(1+\varepsilon))]\}
\end{aligned} \tag{2.15}$$

It remains to show that the rhs of (2.15) is positive for suitable k and ε. This is, however, clearly the case, since ν is a probability measure on $\mathbf{R}$ nonconcentrated on a single point.

The above argument just shows that ν must have a singular continuous component. However, it is easy to show that the measure ν is "pure," i.e., singular continuous. In fact if we define

$$\bar{\Sigma} = \bigcup_L \bigcup_{\omega_L} T_{\omega_L} \Sigma$$

then clearly

$$\nu(\bar{\Sigma}) = 1, \qquad |\bar{\Sigma}| = 0$$

3. PROOF OF THEOREM 2

In this section we compute explicitly the support of the invariant measure for energies of the form $E = 2\cos(\pi q/L)$ and we prove Theorem 2. At the basis of our analysis is the following simple remark: If we consider the composition of n maps T_0

$$T_0^n(x) = P_n(x)/Q_n(x)$$

where we write

$$P_n(x) = a_n x - b_n$$
$$Q_n(x) = c_n x - d_n$$

then we have the recursion relations

$$\begin{aligned} a_{n+1} &= c_n \\ b_{n+1} &= d_n \\ c_{n+1} &= -Ec_n - a_n \\ d_{n+1} &= -Ed_n - b_n \end{aligned} \tag{3.1}$$

with $a_1 = 0$, $b_1 = -1$, $c_1 = -1$, $d_1 = E$. These equations can be solved explicitly for $E \in (-2, 2)$, $E = 2\cos\zeta$, and one obtains

$$\begin{aligned} c_n &= -\frac{\sin n\zeta}{\sin\zeta} = -b_n \\ d_n &= -\frac{\sin(n+1)\zeta}{\sin\zeta} \\ a_n &= -\frac{\sin(n-1)\zeta}{\sin\zeta} \end{aligned} \tag{3.2}$$

Let us now consider the maps

$$T_1 T_0^n = \frac{c_n x - d_n}{(\lambda - E)(c_n x - d_n) - (a_n x - b_n)} \tag{3.3}$$

The equation giving the fixed points (if any) of these maps is

$$c_n x - d_n = \underline{\lambda} c_n x^2 - \underline{\lambda} d_n x - a_n x^2 + b_n x$$

with $\underline{\lambda} = \lambda - E$, that is,

$$x = \frac{\underline{\lambda} d_n \pm [(\underline{\lambda} d_n)^2 - 4d_n(\underline{\lambda} c_n - a_n)]^{1/2}}{2(\underline{\lambda} c_n - a_n)} \tag{3.4}$$

If $\underline{\lambda}\,|d_n| \gg |c_n|$, then

$$\underline{\lambda}^2 d_n^2 - 4d_n \underline{\lambda} c_n + 4d_n a_n > 0$$

that is, we have two real solutions of (3.4)

$$\begin{aligned} x_n^u &= \frac{\lambda d_n}{\underline{\lambda} c_n - a_n} - \frac{1}{\underline{\lambda}} + o\left(\frac{1}{\underline{\lambda}_2}\right) \\ x_n^s &= \frac{1}{\underline{\lambda}} + \frac{1}{\underline{\lambda}^2} \frac{c_n}{d_n} + o\left(\frac{1}{\underline{\lambda}^3}\right) \end{aligned} \tag{3.5}$$

x_n^s is a stable fixed point, in fact:

$$\begin{aligned} \frac{d}{dx} T_1 T_0^n(x)\,|_{x = x_n^s} &= [\underline{\lambda}(c_n x - d_n) - a_n x + b_n]^{-2}\,|_{x = x_n^s} \\ &= \left\{ -\frac{\underline{\lambda} d_n}{2} - \frac{1}{2}\,[\underline{\lambda}^2 d_n^2 - 4d_n(\underline{\lambda} c_n - a_n)]^{1/2} + b_n \right\}^{-2} \end{aligned}$$

which for large λ behaves as

$$\sim (\underline{\lambda} d_n)^{-2} + o(\underline{\lambda}^{-3}) \tag{3.6}$$

Thus, in conclusion, if the critical condition $\underline{\lambda}\,|d_n| \gg |c_n|$ is satisfied, the map $T_1 T_0^n$ becomes hyperbolic with a stable fixed point of order $1/\underline{\lambda}$ and an unstable one of order $o(1)$.

Let us now consider a value of the energy E of the form

$$E = 2\cos[k\pi/(L+1)] \tag{3.7}$$

with k and L prime integers and $k < L+1$.

We have immediately from (3.2) that $T_0^{L+1} = 1$. Moreover, the condition $\underline{\lambda}\,|d_n| \gg |c_n|$ is verified for any $n \leqslant L-1$ and λ sufficiently large depending only on L.

For $n = L$ we have instead:

$$T_1 T_0^L(x) = T_1 T_0^{-1}(x) = x/(\lambda x + 1) \tag{3.8}$$

Thus, in this case the stable and unstable fixed points coincide with the origin and obviously $(T_1 T_0^L)'(x)\,|_{x=0} = 1$.

The graph of the maps $T_1 T_0^n$, $0 \leqslant n \leqslant L$, in the interval $[0, \max_{0 \leqslant n \leqslant L} x_n^s]$ is illustrated in Figs. 1 and 2.

Here $\underline{n}$ is such that $x_{\underline{n}}^s = \max_{0 \leqslant n \leqslant L} x_n^s$, and the concavity of the map $T_1 T_0^{\underline{n}}$ depends on the sign of its unstable fixed point. In the above picture

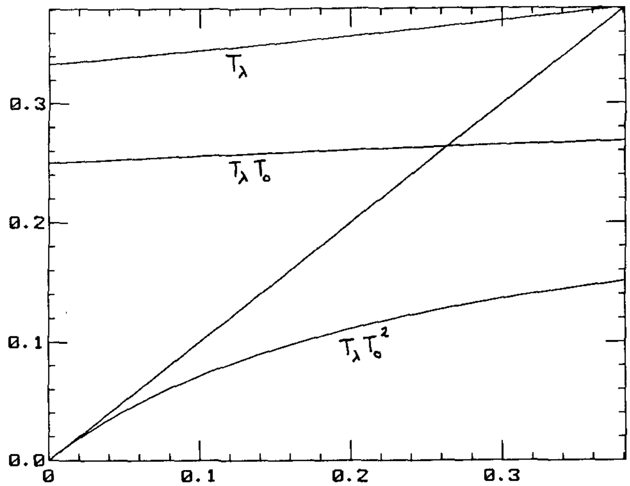

Fig. 1. Graph of the maps $T_1 T_0^n$, $0 \leqslant n \leqslant L$, for $E=1$, $\lambda=4$.

the reader will notice that for any $0 \leqslant i \leqslant L$, $T_1 T_0^{n_i}(x) - T_1 T_0^{n_i-1}(x) > 0$ for any $0 \leqslant x \leqslant x_{\underline{n}}^s$, where $n_1 = L$, $n_L = \underline{n}$. This is in fact true if λ is taken enough, depending only on L, and this, as will appear clear in a moment, is the cause of the Cantor structure of the support of the measure ν. To prove it, we observe that in the interval $[0, x_{\underline{n}}^s]$ all the maps $T_1 T_0^n$, $0 \leqslant n \leqslant L$, are increasing, since their singularity S_n is always $o(1)$ compared with $1/\lambda$.

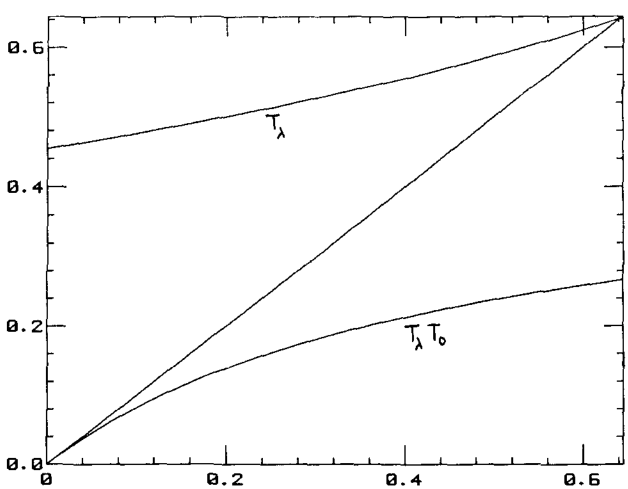

Fig. 2. Graph of the maps $T_1 T_0^n$, $0 \leqslant n \leqslant L$, for $E=0$, $\lambda=2.2$.

Thus, it is enough to show that $T_1 T_0^{n_i}(0) > T_1 T_0^{n_{i-1}}(x_n^s)$ for any $0 \leqslant i \leqslant L$. For $i=1$ we have

$$T_1 T_0^L(x_n^s) = \frac{1/\lambda + c_n/d_n\lambda^2 + o(1/\lambda^3)}{2 + c_n/d_n\lambda + E/\lambda + o(1/\lambda^2)} = \frac{1}{2\lambda} + o\left(\frac{1}{\lambda^2}\right) \tag{3.9}$$

and

$$T_1 T_0^n(0) = T_1 T_0^n(x_n^s) - \int_0^{x_n^s} \frac{d}{dx} T_1 T_0^n(x) \cong \frac{1}{\lambda} + \frac{c_n}{\lambda^2 d_n} + o\left(\frac{1}{\lambda^3}\right)$$

Therefore

$$T_1 T_0^n(0) > T_1 T_0^L(x_n^s) \qquad \text{for any } n < L$$

Analogously, we can prove that

$$T_1 T_0^{n_i}(0) > T_1 T_0^{n_{i-1}}(x_n^s) \qquad \text{for} \quad i = 2, \dots, L \tag{3.10}$$

This easily follows from the following two inequalities:

$$\begin{aligned} x_{n_i}^s - x_{n_{i-1}}^s &\cong \frac{1}{\lambda^2}\left(\frac{c_{n_i}}{d_{n_i}} - \frac{c_{n_{i-1}}}{d_{n_{i-1}}}\right) + o\left(\frac{1}{\lambda^3}\right) \\ &\cong \frac{1}{\lambda^2} \frac{\sin\zeta \sin(n_i - n_{i-1})\,\zeta}{\sin(n_i+1)\,\zeta \sin(n_{i-1}+1)\,\zeta} + o\left(\frac{1}{\lambda^3}\right) \\ &> \frac{1}{\lambda^2} |\sin\zeta \sin j\zeta| + o\left(\frac{1}{\lambda^3}\right), \qquad 1 \leqslant j \leqslant L-1 \\ &> \frac{1}{\lambda^2} \sin^2 \frac{\pi}{L+1} + o\left(\frac{1}{\lambda^3}\right) \end{aligned} \tag{3.11}$$

and for $x \in [0, x_n^s]$

$$\begin{aligned} &|T_1 T_0^n(x) - T_1 T_0^n(x_n^s)| \\ &\quad \leqslant \sup_{x \in [0, x_n^s]} \frac{d}{dx} T_1 T_0^n(x)\, |x - x_n^s| \leqslant \frac{1}{\lambda^2 d_n^2} \frac{1}{\lambda} + o\left(\frac{1}{\lambda^4}\right) \cong \left(\frac{1}{\lambda^3}\right) \end{aligned} \tag{3.12}$$

The first consequence of this analysis of the maps $T_1 T_0^n$ is the following:

Lemma 3. Let

$$\Sigma_0 \equiv [0, x_n^s] \cup \bigcup_{i=1}^{L} T_0^i[0, x_n^s]$$

and let Σ be the support of the invariant measure ν for an energy E of the form (3.7); then

$$\Sigma_0 \supset \Sigma$$

Proof. Σ_0 is an invariant set for the maps T_1 and $T_1 T_0^i$, $1 \leqslant i \leqslant L$, as clearly emerges from the previous discussion (see Fig. 2); furthermore, for any $x \in \mathbf{R}$ there exists a finite sequence of maps $T_{\omega_L(x)}$ such that $T_{\omega_L(x)} \in \Sigma_0$. This clearly implies that with probability one the process z_n will reach the set Σ_0 in a finite number of steps.

Our knowledge of the maps $T_1 T_0^i$ in $[0, x_n^s]$ enables us to state even more than this: let G_i, $i = 0, 1,..., L-1$, be the intervals $[T_1 T_0^{n_i}(x_n^s), T_1 T_0^{n_i+1}(0)]$ (see Fig. 1 for the case $E = 0$) and let

$$\Sigma_0' = \left\{[0, x_n^s] \Big\backslash \left(\bigcup_{j=0}^{L-1} G_j\right)\right\} \cup \bigcup_{i=1}^{L} T_0^i \left\{[0, x_n^s] \Big\backslash \left(\bigcup_{j=0}^{L-1} G_j\right)\right\}$$

Then the same proof of Lemma 3 shows that $\Sigma_0' \supset \Sigma$. It is important at this point to observe that for any $0 \leqslant n, m \leqslant L$, $T_0^n[0, x_n^s] \cap T_0^m[0, x_n^s] = \varnothing$ if λ is large enough.

If we now consider the images of the gaps $\{G_j\}$ under the action of the maps $T_1 T_0^n$, $n = 0, 1,..., L$, then we will obtain new gaps in the set Σ. This is a consequence of the monotonicity of the maps $T_1 T_0^n$ in $[0, x_n^s]$. The Cantor structure of the support Σ will therefore appear by iterating the above argument infinitely many times. To be more precise, let us introduce the set $G^{(n)}$ of gaps produced after n iterations in $[0, x_n^s]$ and let $B^n \equiv [0, x_n^s] \backslash G^{(n)}$. Then clearly

$$G^{(n+1)} = G^{(n)} \cup \left[\bigcup_{i=1}^{L} T_1 T_0^i(G^{(n)})\right]$$

Using again the monotonicity, it is easy to see that $G^{(n+1)}$ consists of the union of the old set of gap $G^{(n)}$ and of a certain number of open intervals mutually disjoint strictly contained in B_n, which represent the new gaps. If we now set

$$B_\infty = \bigcap_{n=1}^{\infty} B_n$$

then by the same proof of Lemma 3 we obtain:

Lemma 4.

$$\Sigma C B_\infty \cup \left[\bigcup_{j=0}^{L} T_1 T_0^j(B_\infty)\right]$$

In order to prove that the invariant measure is singular, we will show that the Lebesgue measure of the Cantor set B_∞ constructed by the previous discussion is zero.

By construction, the set B_∞ satisfies

$$B_\infty = \bigcup_{j=0}^{L-1} T_1 T_0^j(B_\infty) \cup T_1 T_0^L(B_\infty)$$
$$\equiv B^{(<L)} \cup T_1 T_0^L(B_\infty) \tag{3.13}$$

For $i < L$

$$|T_1 T_0^i(B_\infty)| \leqslant \sup_{x \in [0, x_n^s]} (d/dx)\, T_1 T_0^i\, |B_\infty| \equiv A\, |B_\infty|$$

where $|\cdot|$ denotes the Lebesgue measure and $A = o(1/\lambda^2)$. Therefore, we have

$$|B^{(<L)}| \leqslant AL\, |B_\infty| \qquad \text{with } AL \ll 1 \tag{3.14}$$

if $\lambda \gg 1$.

For the map $T_1 T_0^L$ we have

$$\sup_{x \in [0, x_n^s]} \frac{d}{dx} T_1 T_0^L = \frac{d}{dx} T_1 T_0^L(0) = 1$$

and thus we need a more detailed control of $|T_1 T_0^L(B_\infty)|$. Let

$$B_1 \equiv [(T_1\, T_0^L)^2(x_n^s),\, T_1 T_0^L(x_n^s)]$$
$$B_i \equiv T_1 T_0^L(B_{i-1}), \qquad i = 2, 3,... \tag{3.15}$$
$$\underline{B}_i \equiv B_\infty \cap B_i$$

We have

$$\sum_{i=2}^{\infty} |\underline{B}_i| \leqslant C |\underline{B}_1| \tag{3.16}$$

In fact

$$|\underline{B}_{i+1}| = |T_1 T_0^L(\underline{B}_i)| \leqslant \sup_{x \in B_i} (d/dx)\, T_1 T_0^L\, |\underline{B}_i|$$

If we set

$$\alpha_i \equiv \sup_{x \in B_i} (d/dx)\, T_1 T_0^L(x)$$

we get

$$|\underline{B}_{i+1}| \leqslant \prod_{j=1}^{i} \alpha_j \, |\underline{B}_1|$$

and

$$\sum_{i=2}^{\infty} |\underline{B}_i| \leqslant \sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \alpha_j \, |\underline{B}_1|$$

To prove (3.16), we observe that

$$\sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \alpha_j < C$$

$$x_{\underline{n}}^s > \sum_{i=2}^{\infty} |B_i| > \sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \beta_j \, |B_1|$$

where

$$\beta_i \equiv \inf_{x \in B_i} (d/dx) \, T_1 T_0^L(x)$$

and $\beta_i = \alpha_{i-1}$, since

$$\frac{d^2}{dx} T_1 \, T_0^L(x) = -2\lambda(\lambda x + 1)^{-3} < 0$$

in $[0, x_{\underline{n}}^s]$. This implies

$$x_{\underline{n}}^s > \alpha_0 \, |B_1| \left(1 + \sum_{i=2}^{\infty} \prod_{j=1}^{i-1} \alpha_j \right)$$

where

$$\alpha_i \equiv (d/dx) \, T_1 T_0^L[T_1 T_0^L(x_{\underline{n}}^s)]$$

By taking

$$C = x_{\underline{n}}^s / \alpha_0 B_1 = o(1)$$

we get (3.16).

From (3.16) we now get

$$|T_1 T_0^i(B_\infty)| = \sum_{i=1}^{\infty} |\underline{B}_i| \leqslant (C+1) \, |\underline{B}_1| \tag{3.17}$$

Since

$$\underline{B}_1 \equiv T_1 T_0^L(B^{(<L)})$$
$$|\underline{B}_1| \leqslant \frac{d}{dx} T_1 T_0^L \Big|_{x=(T_1 T_0)^2(x_n^s)} |B^{(<L)}| = \alpha_1 |B^{(<L)}| \tag{3.18}$$

From (3.14) and (3.17) we conclude that

$$|B_\infty| \leqslant |B_\infty| AL[1 + (C+1)\alpha_1]$$

with $A \cong o(1/\lambda^2)$; for λ sufficiently large, this implies $|B_\infty| = 0$.

Let us now examine the special case $E = 0$ and $\lambda = 2$. Then the two fixed points of the map T_1 collapse together into the point $x = 1$ and in order to determine the structure of the support of the invariant measure in the set (0, 1) we will have to consider only two maps:

$$T_1(x) = 1/(2-x), \qquad T = T_1 \circ T_0(x) = x/(2x+1)$$

Clearly the interval (1/3, 1/2) is a gap and therefore the support of the invariant measure inside (0, 1) is again a Cantor set. It remains to check that its measure is zero and we will do that by redoing the previous computation in a slightly more careful way. The reason for that is that the point $x = 1$ is no longer a stable point for the map T_1 and $T_1'(x=1) = 1$.

Let us denote by A and B the parts of the support S inside the intervals ((1/2, 1) and (0, 1/3), respectively. Then we have

$$A = T_1(A) \cup T_1(B), \qquad B = T(A) \cup T(B) \tag{3.19}$$

By iterating (3.19) infinitely often, we get that the Lebesgue measure of A, $|A|$, is bounded by

$$|A| < |T_1(A)| + \sup_{0<x<1/3} T_1'(x) \sum_{n=1} |T^n(A)| \tag{3.20}$$

Since $T^n(x) = x/(2nx+1)$, we get from (3.20) the inequality

$$\begin{aligned} |A| &< |T_1(A)| + (9{,}25) \sum_{n=1} (1/n^2)\,|A| \\ &= |T_1(A)| + (27/50)\,|A| \end{aligned} \tag{3.21}$$

Let us now write A as $A = \bigcup_i A_i$, where, $A_{i+1} = T_1(A_i) = T_1^{i-1}(A_1)$ and $A_1 = A(1/2, T_1(1/2))$. Using the explicit formula for $T_1^{i-1}(x) = 1 - [i - 1 + 1/(1-x)]^{-1}$, it is easy to check that the following inequality holds:

$$\sum_{n=2} |A_n| < |A_1| \sum_{n=1} 25/(2n+5)^2 \tag{3.22}$$

which, combined with (3.21), gives

$$|A_1| < |A_1| \left\{(27/50)\left[\sum_{n=0} 25/(2n+5)^2\right]\right\} \tag{3.23}$$

A little estimate shows now that the numerical factor appearing in the rhs of (3.23) is smaller than 1, thus proving that $|A_1|$ and *a fortiori* $|A|$ and $|B|$ must vanish.

We now check that indeed for the case under consideration the Liapunov exponent *does not* satisfied the inequality required in Theorem 1. To do that, we used the following elementary upper bound on the Liapunov exponent[1]:

$$\gamma \leqslant 1/2 \log |\Lambda| \tag{3.24}$$

where Λ is the largest eigenvalue of the 4×4 matrix A defined as follows:

$$A = \begin{bmatrix} 4(1-p) & -4(1-p) & 0 & 1 \\ 2(1-p) & -1 & 0 & 0 \\ 2(1-p) & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Using the explicit form of the matrix A, it is very easy to see that its largest eigenvalue Λ is strictly less than 2 in absolute value for any value of p in (0, 1).

4. MULTIFRACTAL STRUCTURE OF THE INVARIANT MEASURE AT $E=0$

In this section we investigate, mainly numerically, more deeply the structure of the support of the invariant measure for $E=0$ and $\lambda > 2$. For these values of the parameters we found in Section 2 that the part of the support contained in $[0, X_s]$, X_s being the stable fixed point of the map T_1, is a Cantor set Σ and it is therefore a natural question to ask whether ν and Σ have a multifractal structure. By this we mean the following: let $\varepsilon > 0$ and for $x \in \Sigma$ let $B_\varepsilon(x)$ be the interval of length ε centered at x. Then one expects that the ν-measure of $B_\varepsilon(x)$ will scale as ε goes to zero as

$$\nu(B_\varepsilon(x)) \simeq \varepsilon^{\alpha(x)}$$

If the scaling exponent $\alpha(x)$ attains different values on different sets of x's, then we say that ν is multifractal (see, e.g., Ref. 13 for a nice review on this subject). It is, however, important to recognize that in general for ν-almost all x in Σ, $\alpha(x)$ is independent of x, $\alpha(x) = \alpha^0$ (see Theorem 5.1 of Ref. 11). Values of α different from α^0, say $\alpha(x) = \alpha$, are attained over subsets of Σ of Hausdorff dimension $f(\alpha)$ smaller than that of Σ.

These and related topics are discussed in an interesting paper,[12] which stimulated subsequent work (see also Ref. 14 for a critical discussion). In particular, in Ref. 15 all the statements that will follow have been proved rigorously for the invariant measure of expanding Markov maps on the unit interval.

In order to compute the scalings of the measure ν, we first observe that it follows in a trivial way that the restriction of ν to the interval $[0, x_s]$ coincides, apart from a normalization factor, with the invariant measure μ of the Markov chain in $[0, x_s]$ generated by taking the maps T and T_1 restricted to $[0, x_s]$ with probability $p' = p/(1+p)$ and $1 - p' = 1/(1+p)$, respectively. Then the recipe to compute the dimension $f(\alpha)$ of the set of singularity of strength α for μ is the following:

For $q, t \in R$, let $F(q, t)$ be given by

$$F(q, t) = \lim_{L \to \infty} \sup_x \frac{1}{L} \ln \sum_{\omega_L} \frac{p^q_{\omega_L}}{|T_{\omega_L}[0, x_s]|^t} \tag{4.1}$$

and let $t(q)$ be the (unique) value of t such that $F(q, t(q)) = 0$. It is easily seen that $t(q)$ is a concave function of q, and its Legendre transform

$$f(\alpha) = \min_q \{q\alpha - t(q)\} \tag{4.2}$$

gives the Hausdorff dimension of the set, where $\alpha(x) = \alpha$, provided that $\alpha_{\min} < \alpha < \alpha_{\max}$, where $\alpha_{\min(\max)} = \lim_{q \to +(-)\infty} t(q)$. Furthermore, $f(\alpha)$ is concave with a unique maximum at α^* and $f(\alpha^*)$ coincides with the Hausdorff dimension of the set Σ. In our context the existence of the function $F(q, t)$ is guaranteed by the following proposition (see the Appendix for a proof):

Proposition 1. (a) $F(q, t)$ exists for any q, t and it is jointly continuous in t, q and strictly decreasing (respectively increasing) in q (t).

(b) There exists a unique continuous, concave, strictly increasing function $t(q)$ such that $F(q, t(q)) = 0$ for any q.

We did not attempt to prove the statement concerning the function $f(\alpha)$, but we believe that the ideas in Ref. 15 should apply also in this case. Instead, we computed numerically $t(q)$ and its Legendre transform $f(\alpha)$ by exploring the contribution to $F(q, t)$ of trajectories with length $L = 8$. The results for $\lambda = 2.2$ and $p' = 0.3$ are given in Fig. 3 and show a continuous distribution of scalings α between the values $\alpha_{\min} = 0.5$ and $\alpha_{\max} = 1.2$. The

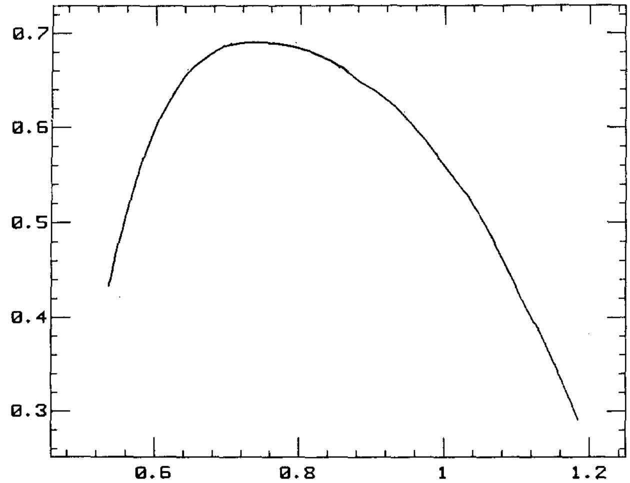

Fig. 3. Graph of $f(\alpha)$ for $\lambda = 2.2$ and $p' = 0.3$.

"typical" scaling is $\alpha^* = 0.75$, to which corresponds a value $f(\alpha^*) = 0.69$. Figure 4 plots instead the graph of the function $D(q) = t(q)/(q-1)$. It is important to observe at this point that it is quite difficult to obtain reasonable numerical results for finite L (e.g., $L = 10$) and negative q and t (of order -20) if the parameter p' is greater than 0.5. The reason is that for these values of the parameters the dominant trajectories in the sum in (4.1) are those with very few maps T_1 and for such trajectories the small intervals $|T_{\omega_L}[0, x_s]|$ do not scale exponentially in L, since $T'(0) = 1$.

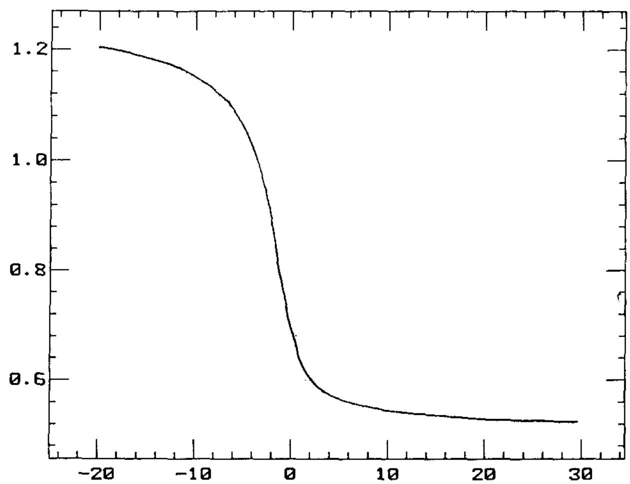

Fig. 4. Graph of $D(q)$ for $\lambda = 2.2$ and $p' = 0.3$.

APPENDIX. PROOF OF PROPOSITION 3

We split the proof into several steps.

1. We have

$$\lim_{L\to\infty} \sup_x \frac{1}{L} \left| \ln \left\{ \sum_{\omega_L} \frac{p^q_{\omega_L}}{|T_{\omega_L}[0, x_s]|^t} \Big/ \sum_{\omega_L} \frac{p^q_{\omega_L}}{[T'_{\omega_L}(x)]^t} \right\} \right| = 0 \tag{A.1}$$

In fact

$$\begin{aligned} \sum_{\omega_L} \frac{p^q_{\omega_L}}{|T_{\omega_L}[0, x_s]|^t} \Big/ \sum_{\omega_L} \frac{p^q_{\omega_L}}{[T'_{\omega_L}(x)]^t} &< \max_{\omega_L} \left(\frac{T'_{\omega_L}(x)}{|T_{\omega_L}[0, x_s]|} \right)^t \\ &> \min_{\omega_L} \left(\frac{T'_{\omega_L}(x)}{|T_{\omega_L}[0, x_s]|} \right)^t \end{aligned} \tag{A.2}$$

and

$$\frac{T'_{\omega_L}(x)}{|T_{\omega_L}[0, x_s]|} = \frac{T'_{\omega_L}(x)}{T'_{\omega_L}(\xi)} \frac{1}{x_s}, \qquad 0 \leqslant \xi \leqslant x_s$$

with $\xi = \xi(\omega_L)$.

We can now write

$$\frac{T'_{\omega_L}(x)}{T'_{\omega_L}(\xi)} = \prod_{i=1}^{L} \frac{T'_{\omega_{L(i)}}(x_i)}{T'_{\omega_{L(i)}}(\xi_i)}, \qquad x_i = T_{\omega_{i-1}}(x)$$

and analogously for ξ. Now

$$\begin{aligned} \frac{T'_{\omega_{L(i)}}(x_i)}{T'_{\omega_{L(i)}}(\xi_i)} &= 1 + \frac{T''_{\omega_{L(i)}}(\eta)}{T'_{\omega_{L(i)}}(\xi_i)} (x_i - \xi_i) \\ &\geqslant 1 + K_1\, |x_i - \xi_i| \\ &\geqslant 1 - K_2\, |x_i - \xi_i| \end{aligned} \tag{A.3}$$

It remains to show that $x_i - \xi_i$ goes to zero as $i \to \infty$. By explicit computation we get

$$|x_i - \xi_i| \leqslant (\beta \cdot i + 1)^{-1} \qquad \forall \omega_L \tag{A.4}$$

for a suitable constant β.

This estimate is very bad for a "typical" ω_L, but it is optimal for ω_L's with a large number of maps T. By plugging (A.4) into (A.3), we get (A.1).

2. Let

$$\underline{F}_L(t, q) = \sup_x \ln \sum_{\omega_L} \frac{p^q_{\omega_L}}{[T'_{\omega_L}(x)]^t}$$

Then

$$\underline{F}_{L+k} \leqslant \underline{F}_L + \underline{F}_k \tag{A.5}$$

Subadditivity is just a consequence of the chain rule:

$$T'_{\omega_{L+k}}(x) = T'_{\omega_L}(T_{\omega_k}(x)) \cdot T'_{\omega_k}(x) \tag{A.6}$$

Thus

$$\begin{aligned}\underline{F}_{L+k} &= \sup_x \ln \sum_{\omega_k} \left(\frac{p^q_{\omega_k}}{T'^t_{\omega_k}(x)} \sum_{\omega_L} \frac{p^q_{\omega_L}}{T'_{\omega_L}[T_{\omega_k}(x)]^t} \right) \\ &\leqslant \sup_x \ln \sum_{\omega_k} \frac{p^q_{\omega_k}}{[T'_{\omega_k}(x)]^t} + \sup_x \ln \sum_{\omega_L} \frac{p^q_{\omega_L}}{[T'_{\omega_L}(x)]^t} \\ &= \underline{F}_L + \underline{F}_k\end{aligned}$$

Using 1 and 2, we get the existence of the function $F(q, t)$. Joint continuity of $F(q, t)$ in q, t follows by straigthforward estimates.

To prove strict monotonicity of $F(q, t)$ as a function of t, we need the following result:

3. We have

$$\begin{aligned}F(q, t) &= \lim_{L \to \infty} \sup_x \frac{1}{L} \ln \sum_{\omega_L} \frac{p^q_{\omega_L}}{|T_{\omega_L}[0, x_s]|^t} \\ &= \lim_{L \to \infty} \sup_x \frac{1}{L} \ln \sum_{\omega_L}^{(>n)} \frac{p^q_{\omega_L}}{|T_{\omega_L}[0, x_s]|^t}\end{aligned}$$

with $n = \alpha L$ and $\alpha \ll 1$, where $\sum^{(>n)}$ is the sum over all ω_L with more than n maps T_1. In order to prove this identity, we use the following remark: Let ω_L be a trajectory with n maps T_1; we denote by k the number of bloks of maps T of length m_i, $i = 1,..., k$, that is, $\sum_1^k m_i = L - n$; then

$$\theta \equiv |T_{\omega_L}[0, x_s]| \geqslant \exp(-\beta n - 2cn) \exp[-L \exp(-c)] \tag{A.7}$$

where $e^{-\beta} = \inf_x T'_\lambda$ and c is a suitable constant.

In fact, since $T^m(x) = x/(\lambda m x + 1)$, and therefore

$$[T^m(x)]' = (\lambda m x + 1)^{-2} \geqslant (\lambda m x_s + 1)^{-2} \geqslant D/m^2$$

for D sufficiently small, we have

$$\theta \geqslant e^{-\beta n} \prod_{1}^{k} D/m_i^2 = e^{-\beta n} e^{-k \ln(1/D)} \prod_{1}^{k} 1/m_i^2$$

By using the Jensen inequality and the obvious remark $k \leqslant n+1$, we have

$$\ln\left(\prod_{1}^{k} m_i^2\right) = 2k \sum_{1}^{k} (1/k) \ln m_i \leqslant 2k \ln\left[\sum_{1}^{k} (1/k)\, m_i\right]$$
$$= 2k \ln[(L-n)/k] \leqslant 3k \ln(L/k)$$

By studying the function $x \ln(L/x)$ it is easy to see that

$$k \ln(L/k) \leqslant ck + Le^{-c}$$

for any $c > 0$, which proves (A.7).

Now we come back to the proof of identity 3. The sum $\sum_{\omega_L}$ appearing in the definition of $F(q, t)$ is split into two sums $\sum_{\omega_L}^{(<\alpha L)}$ and $\sum_{\omega_L}^{(>\alpha L)}$ where, as before, in the first sum we consider only the trajectories with at most αL maps T_1. By means of the estimate (A.7) together with the trivial one

$$|T_{\omega L}[0, x_s]| \leqslant e^{-\beta' n} \tag{A.8}$$

where n is the number of maps T_1 in T_{ω_L} and $e^{-\beta'} = \inf_x T'_{\omega_L}(x)$, it is easy to see that it is possible to choose the constant c in (A.7) sufficiently large and α sufficiently small in such a way that $\sum_{\omega_L}^{(<\alpha L)} < \sum_{\omega_L}^{(>\alpha L)}$ for any L sufficiently large. This shows that

$$F(q, t) = \lim_{L \to \infty} (1/L) \ln \sum_{\omega_L}^{(>\alpha L)}$$

Once this has been established, it is easy to see, using again (A.8), that $F(q, t+\delta) > \alpha\delta + F(q, t)$. Strict monotonicity in q follows by direct inspection as well as the other statements of the proposition.

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